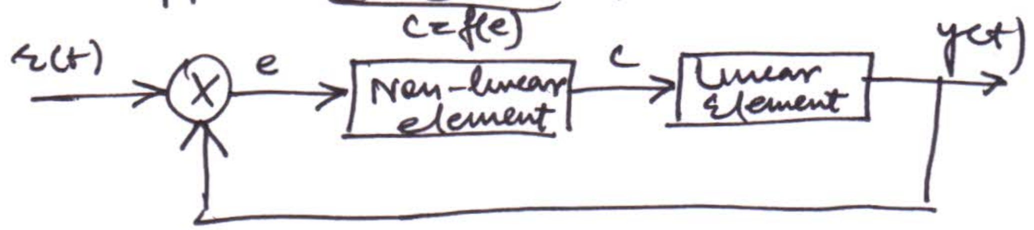
Selected Topics in Advanced Control
(9th 9 ?)
(Dr M. Ahmed)

Describing Function Analysis

→ Application Domain:

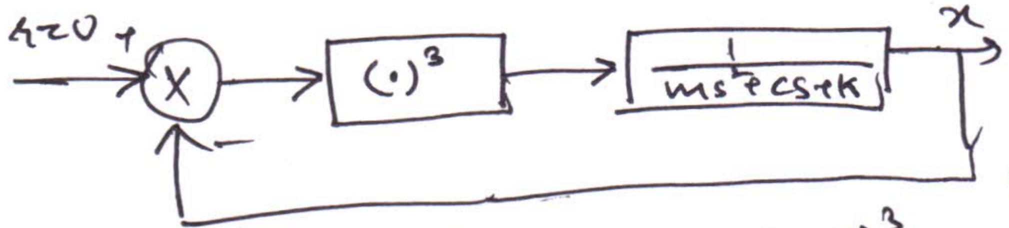


→ Almost linear system
→ One hard non-linearity, otherwise linear

→ Genuinely non-linear whose dynamic Equations can be arranged in the form of [illegible].

e.g. $m\ddot{x} + c\dot{x} + kx + k_1x^3 = 0$

$m\ddot{x} + c\dot{x} + kx = -k_1x^3$



$x = \frac{1}{ms^2+cs+k} \times (-x)^3$

$ms^2x + \cancel{ksx} + kx = -k_1x^3$

$m\ddot{x} + c\dot{x} + kx = -k_1x^3$

$c(t) = c_1(t)$
$= a_1\cos\omega t + b_1\sin\omega t$
$= M\sin(\omega t + \varphi)$
where $M(A,\omega) = \sqrt{a_1^2 + b_1^2}$
$\varphi(A,\omega) = \tan^{-1} a_1/b_1$

$c(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\omega t + b_n\sin n\omega t$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} c(t)\, d\omega t$

Basic Assumptions:

→ Single non-linear component
→ Non-linear component is time-invariant
→ Corresponding to a sinusoidal i/p $e = \sin\omega t$, only the fundamental component in the o/p $c(t)$ has to be considered.
→ Non-linearity is odd.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} c(t)\cos(n\omega t)\, d\omega t$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} c(t)\sin(n\omega t)\, d\omega t$

DESCRIBING FUNCTIONS (CONTD.)

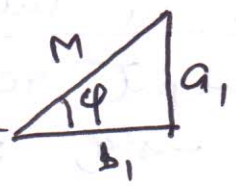
$$A\sin\omega t \longrightarrow \boxed{\text{NON-LINEAR ELEMENT}} \xrightarrow{c(t)}$$

FUNDAMENTAL COMPONENT OF $c(t)$,

$c_1(t) = a_1 \cos\omega t + b_1 \sin\omega t$ (Assuming the non-linearity to be such that $a_0 = 0$)

Put $a_1 = M\sin\varphi$ and $b_1 = M\cos\varphi$

$M = \sqrt{a_1^2 + b_1^2}$ & $\varphi = \tan^{-1}\dfrac{a_1}{b_1}$



$\therefore c_1(t) = M\sin\varphi\cos\omega t + M\cos\varphi\sin\omega t$

$= M\sin(\omega t + \varphi)$

$$\begin{matrix} A\sin\omega t \\ \text{or } Ae^{j\omega t} \end{matrix} \longrightarrow \underset{N(A,\omega)}{\boxed{\text{NON-LINEAR ELEMENT}}} \longrightarrow \begin{matrix} M\sin(\omega t+\varphi) \\ Me^{j(\omega t+\varphi)} \end{matrix}$$

$$N(A,\omega) = \frac{Me^{j(\omega t+\varphi)}}{Ae^{j\omega t}} = \frac{M}{A}e^{j\varphi}$$

where $N(A,\omega)$ represents the describing function of the non-linear element.

a_1 & b_1, as has been seen, can be evaluated as:

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} c(t)\cos\omega t \, d\omega t \quad \&$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} c(t)\sin\omega t \, d\omega t$$

COMPUTING DESCRIBING FUNCTIONS:

1. ANALYTICAL CALCULATION: When the non-linear characteristics $c = f(e)$, where e is the i/p and c is the output, of the non-linear element are described by an explicit function, then analytical evaluation of a_1 & b_1 is desirable. The explicit

function $f(e)$ of the non-linear element may be an idealized representation of the simple non-linearities such as saturation and dead zone, or it may be a curve-fit of the i/p - o/p relationship of an element. However, for non-linear elements which evade convenient analytical expressions, the analytical technique is difficult.

2. Numerical integration: For non-linearities whose input-output relationship $c = f(e)$ is given by graphs or tables, it is convenient to use numerical integration to evaluate the describing function. The idea is to approximate integrals by discrete sums over small intervals. Various numerical integration techniques can be used for this purpose. It is obviously important that the numerical integration can be easily implemented by computer programs. The result is a plot representing the describing function.

3. Experimental Evaluation: The experimental method is particularly suitable for complex non-linearities and dynamic non-linearities. When a system non-linearity can be isolated and excited with sinusoidal inputs of known amplitude and frequency, experimental determination of the describing function can be done by using a harmonic analyses on the o/p of the non-linear element. This is quite similar to the determination of frequency response of linear elements experimentally. The difference here is that not only the frequencies, but also the amplitude of the input

sinusoid should be varied. The result of the experiments are a set of curves in the complex plane representing the describing function $N(A,\omega)$, instead of analytical expressions. Specialized instruments are available which automatically compute the describing functions of non-linear elements based on the measurement of non-linear element response to harmonic excitation.

Ex: The characteristics of a non-linear spring are given by:

$$y = x + x^3/2$$

Compute its describing function, with x being the i/p and y being the o/p.

Soln: Given an i/p $x(t) = A \sin \omega t$, we have

$$y(t) = A \sin \omega t + \frac{A^3 \sin^3 \omega t}{2}$$

The o/p can be expanded by Taylor series expansion with the fundamental component given by:

$$y_1(t) = a_0 + a_1 \cos \omega t + b_1 \sin \omega t$$

Since $y(t)$ is an odd function, therefore

$$a_0 = a_1 = 0$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[A \sin \omega t + \frac{A^3}{2} \sin^3 \omega t\right] \sin \omega t \, d\omega t$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} A \sin^2 \omega t \, d\omega t + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{A^3}{2} \sin^4 \omega t \, d\omega t$$

$$b_1 = \frac{A}{\pi} \times \pi + \frac{1}{\pi} \frac{A^3}{2} \cdot \frac{3\pi}{4}$$

$$= A + \frac{3}{8} A^3$$