Department	Mechanical		Program	M. Tech		
Subject Name	FEM		Subject Code	MSD 102		
Semester	IST	Credits	4	Teacher Incharge/Mentor		Prof. N. A. Sheikh
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Unit I:

Links to the Resources:

1. https://www.youtube.com/watch?v=KR74TQesUoQ&list=PLbMVogVj5nJRjnZA9oryBmDdUNe7lbnB0

2. http://nptel.ac.in/courses/105108141/

3. https://www.youtube.com/watch?v=URbiADhc_rA

4. https://www.youtube.com/watch?v=U65GK1vVw4o&list=PLJhG_d-Sp_JHKVRhfTgDqbic_4MHpltXZ

Books to be Consulted:

1. Segerlind L.J, "Applied Finite Element Analysis", *John Wiley Publishers, Second Edition, 1976.*
2. Bathe, K.J. and Wilson, E.L., "Numerical Methods in Finite Element Analysis", *Prentice Hall,1976.*
3. J. N. Reddy, An Introduction to the Finite Element Method, 3rd Edition.
4. Tirupathy R. Chandrupatla and Ashok D. Belegundu, Introduction to Finite Elements in Engineering, 3rd Edition.

Lecture Notes:

Historical Background

The words "finite element method" were first used by Clough in his paper in the Proceedings of 2nd ASCE (American Society of Civil Engineering) conference on Electronic Computation in 1960. Clough extended the matrix method of structural analysis, used essentially for frame-like structures, to two-dimensional continuum domains by dividing the domain into triangular elements and obtaining the stiffness matrices of these elements from the strain energy expressions by assuming a linear variation for the displacements over the element. Clough called this method as the finite element method because the domain was divided into elements of finite size. (An element of infinitesimal size is used when a physical statement of some balance law needs to be converted into a mathematical equation, usually a differential equation).

Argyris, around the same time, developed similar technique in Germany. But, the idea of dividing the domain into a number of finite elements for the purpose of structural analysis is older. It was first used by Courant in 1943 while solving the problem of the torsion of non-circular shafts. Courant used the integral form of the balance law, namely the expression for the total potential energy instead of the differential form (i.e., the equilibrium equation). He divided the shaft cross-section into triangular elements and assumed a linear variation for the primary variable (i.e., the stress function) over the domain. The unknown constants in the linear variation were obtained by minimizing the total potential energy expression. The Courant's technique is called as applied mathematician's version of FEM where as that of Clough and Argyris is called as engineer's version of FEM.

From 1960 to 1975, the FEM was developed in the following directions:

(1) FEM was extended from a static, small deformation, elastic problems to dynamic (i.e., vibration and transient) problems, small deformation fracture, contact and elastic -plastic problems, non-structural problems like fluid flow and heat transfer problems.

(2) In structural problems, the integral form of the balance law namely the total potential energy expression is used to develop the finite element equations. For solving non-structural problems like the fluid flow and heat transfer problems, the integral form of the balance law was developed using the weighted residual method.

(3) FEM packages like NASTRAN, ANSYS, and ABAQUS etc. were developed.

The large deformation (i.e., geometrically non-linear) structural problems, where the domain changes significantly, were solved by FEM only around 1976 using the updated Lagrangian formulation. This technique was soon extended to other problems containing geometric non-linearity :

dynamic problems,

fracture problems,

contact problems,

elastic-plastic (i.e., materially non-linear) problems.

Basic Steps

The finite element method involves the following steps.

First, the governing differential equation of the problem is converted into an integral form. There are two techniques to achieve this: (i) Variational Technique and (ii) Weighted Residual Technique. In variational technique, the calculus of variation is used to obtain the integral form corresponding to the given differential equation. This integral needs to be minimized to obtain the solution of the problem. For structural mechanics problems, the integral form turns out to be the expression for the total potential energy of the structure. In weighted residual technique, the integral form is constructed as a weighted integral of the governing differential equation where the weight functions are known and arbitrary except that they satisfy certain boundary conditions. To reduce the continuity requirement of the solution, this integral form is often modified using the divergence theorem. This integral form is set to zero to obtain the solution of the problem. For structural mechanics problems, if the weight function is considered as the virtual displacement, then the integral form becomes the expression of the virtual work of the structure.

In the second step, the domain of the problem is divided into a number of parts, called as elements. For one-dimensional (1-D) problems, the elements are nothing but line segments having only length and no shape. For problems of higher dimensions, the elements have both the shape and size. For two-dimensional (2D) or axi-symmetric problems, the elements used are triangles, rectangles and quadrilateral having straight or curved boundaries. Curved sided elements are good choice when the domain boundary is curved. For three-dimensional (3-D) problems, the shapes used are tetrahedron and parallelepiped having straight or curved surfaces. Division of the domain into elements is called a mesh.

In this step, over a typical element, a suitable approximation is chosen for the primary variable of the problem using interpolation functions (also called as shape functions) and the unknown values of the primary variable at some pre-selected points of the element, called as the nodes. Usually polynomials are chosen as the shape functions. For 1-D elements, there are at least 2 nodes placed at the end-points. Additional nodes are placed in the interior of the element. For 2-D and 3-D elements, the nodes are placed at the vertices (minimum 3 nodes for triangles, minimum 4 nodes for rectangles, quadrilaterals and tetrahedral and minimum 8 nodes for parallelepiped shaped elements). Additional nodes are placed either on the boundaries or in the interior. The values of the primary variable at the nodes are called as the degrees of freedom.

To get the exact solution, the expression for the primary variable must contain a complete set of polynomials (i.e., infinite terms) or if it contains only the finite number of terms, then the number of elements must be infinite. In either case, it results into an infinite set of algebraic equations. To make the problem tractable, only a finite number of elements and an expression with only finite number of terms are used. Then, we get only an approximate solution. (Therefore, the expression for the primary variable chosen to obtain an approximate solution is called an approximation). The accuracy of the approximate solution, however, can be improved either by increasing the number of terms in the approximation or the number of elements.

In the fourth step, the approximation for the primary variable is substituted into the integral form. If the integral form is of variational type, it is minimized to get the algebraic equations for the unknown nodal values of the primary variable. If the integral form is of the weighted residual type, it is set to zero to obtain the algebraic equations. In each case, the algebraic equations are obtained element wise first (called as the element equations) and then they are assembled over all the elements to obtain the algebraic equations for the whole domain (called as the global equations).

In this step, the algebraic equations are modified to take care of the boundary conditions on the primary variable. The modified algebraic equations are solved to find the nodal values of the primary variable.

In the last step, the post-processing of the solution is done. That is, first the secondary variables of the problem are calculated from the solution. Then, the nodal values of the primary and secondary variables are used to construct their graphical variation over the domain either in the form of graphs (for 1-D problems) or 2-D/3-D contours as the case may be.

Advantages of the finite element method over other numerical methods are as follows:

The method can be used for any irregular-shaped domain and all types of boundary conditions.

Domains consisting of more than one material can be easily analyzed.

Accuracy of the solution can be improved either by proper refinement of the mesh or by choosing approximation of higher degree polynomials.

The algebraic equations can be easily generated and solved on a computer. In fact, a general purpose code can be developed for the analysis of a large class of problems.

Integral Formulations:

In this lecture, integral formulations of a boundary value problem are developed. There are two types of integral formulations:

- Weak or Weighted residual formulation and
- Variational formulation

In finite element method, the solution of a boundary value problem is obtained by using one of these two integral formulations. When it is difficult to solve the differential equation of a boundary value problem, this method provides an alternative way to obtain the solution. But, usually, it is an approximate solution

Model Boundary Value Problem

To illustrate the development of integral formulations, the following model boundary value problem is considered. It represents the axial extension (or compression) of a bar shown in Fig. 1.

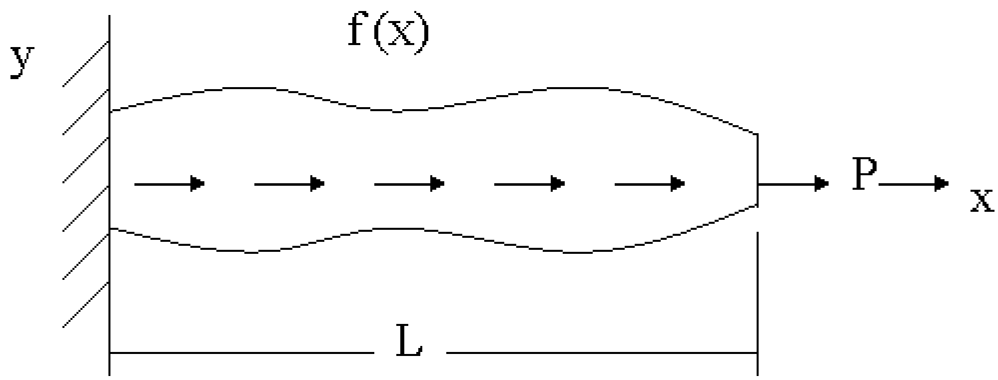

Figure 1

The bar has a variable area of cross-section which is denoted by the function *A*(*x*). The length of the bar is *L*. The Young's modules of the bar material is *E* . The bar is fixed at the end *x* = 0. The forces acting on the bar are (i) a distributed force *f*(*x*), which varies with *x* and (ii) a point force *P* at the end *x* = *L* . The axial displacement of a cross-section at *x*, denoted by *u*(*x*), is governed by the following boundary value problem consisting of a differential equation (DE) and two boundary conditions (BC):

DE: $-\frac{d}{dx}(EA(x)\frac{du}{dx}) = f(x)$ $\quad 0<x<L$ (1a)

BC: (i) u = 0 $\quad$ at x=0 (1b)

(ii) EA(x) $\frac{du}{dx}$ $\quad$ at x=L (1c)

The differential equation represents the equilibrium of a small element of the bar expressed in terms of the displacement using the stress-strain and strain-displacement relations. The boundary condition (1b) is a geometric or kinematic boundary condition. Since, it is a condition on the primary variable *u*(*x*), it is called as **Dirichlet boundary condition**. The second boundary condition (condition 1c) is a force boundary condition, or a condition on the secondary variable (i.e., axial force). Since, it is a condition on a derivative of the primary variable; it is called as the **Neumann boundary condition**.

Weak or Weighted Residual Formulation

Consider a function *u* (*x*), defined over the interval [0, *L*], which satisfies both the boundary conditions (1b) and (1c) but otherwise arbitrary. In general, such a function will not satisfy the differential equation (1a). It means, when *u* (*x*) is substituted in the left hand side of equation (1a), it will not be equal to *f* (*x*). In this case, the difference is called as residue or error and is denoted by *R* (*x*). Thus,

$$R(x) = f(x) - (-\frac{d}{dx}(EA(x)\frac{du}{dx})) \qquad (1.2)$$

In Weighted Residual Formulation, an approximate solution to the problem (1a, 1b and 1c) is obtained by minimizing the 'weighted' residue or the product of the residue *R*(*x*) and certain weight function, denoted by *w*(*x*).

The weight function is chosen to be an arbitrary function except that it is required to satisfy the following conditions:

1. At the boundary where u is specified, w must be zero. Thus, in the present problem, $w = 0$ at $x = 0$.
2. At the boundary, where the derivative of u is specified, w must be unconstrained. Thus in the present problem, w is unconstrained at $x = L$.
3. The function w should be smooth enough for the integral of the weighted residue to be finite.

A collection of all the functions, which satisfy the above conditions, is called as a set or class of admissible functions. Thus, the weight function must belong to the class of admissible functions.

One common way of minimizing the residue $R(x)$ is to set the integral of the product of $R(x)$ and $w(x)$ to zero for any admissible function $w(x)$. Thus, an approximate solution to the problem (1a,1b,1c) is obtained from the following equation:

$$\int_0^L [f(x) - (-\frac{d}{dx}(EA\frac{du}{dx}))]w(x)dx = 0 \tag{1.3}$$

or,

$$\int_0^L -\frac{d}{dx}(EA\frac{du}{dx})wdx = \int_0^L fwdx \tag{1.4}$$

for any w belonging to the class of admissible functions. To relax the smoothness requirements on the choice of the approximate function $u(x)$, the left side of equation (2.4) is usually integrated by parts. This makes the expression symmetric in u and w . By carrying out the integration by parts, equation (1.4) becomes

$$\int_0^L EA\frac{du}{dx}\frac{dw}{dx}dx = \int_0^L fwdx + EA\frac{du}{dx}w\Big|_{x=L} - EA\frac{du}{dx}w\Big|_{x=0} \tag{1.5a}$$

Since w is zero at $x = 0$, the last term is zero. Further, u satisfies the boundary condition (1.1c). Then EA (du/dx) at $x = L$ becomes equal to P . With these simplifications, equation (1.5a) becomes:

$$\int_0^L EA\frac{du}{dx}\frac{dw}{dx}dx = \int_0^L fwdx + (Pw)\Big|_{x=L} \tag{1.5b}$$

This is called as the weighted residual integral. This is the integral form used in the weighted Residual Formulation. Now the condition 3 of the class of the admissible functions can be made explicit. For all the integrals of equation (1.5b) to be finite, dw/dx must be finite at every point of the interval (0, L). It means dw/dx must be piecewise continuous on (0, L) with only finite discontinuities.

Expression (1.5b) is also called as the **Weak Formulation** of the boundary value problem (1.1a), (1.1b) and (1.1c) because the solution given by the formulation is required to satisfy weaker smoothness conditions compared to that of the solution of the original boundary value problem.

Depending on the choice of w , various special forms of the weighted residual method exist. They are :

(i) Galerkin Method,
(ii) Petrov Galerkin Method, and
(iii) Least Square Method.

If the condition of smoothness on w (i.e. the 3rd condition) is relaxed, two more special forms emerge:

(i) Sub-domain Method and
(ii) Collocation Method.

Galerkin Method

$$\int_0^L EA\frac{du}{dx}\frac{dw}{dx}dx = \int_0^L fw dx + Pw\,/_{x=L} \qquad (1.6)$$

- We assume an approximate solution as a series of (N +1) terms :

$$u^{N+1}(x) \equiv u_h(x) \equiv u_{FE}(x) = \sum_{j=1}^{N+1} u_j\phi_j(x) \qquad (1.7)$$

- where u_j are the unknown coefficients and $\phi_j(x)$ are the basis functions described in the previous section. Note that, now the basis functions are numbered from one rather than from zero. Therefore the summation index also starts from one, and not from zero.
- For Galerkin method, we choose the basis functions $\phi_i(x)$ as the weight functions. Thus,

$$w(x) = \phi_i(x) \qquad \text{for } i=1,2,...,N+1 \qquad (1.8)$$

- Note that these functions are linearly independent and satisfy all the constraints arising out of the three admissibility conditions on $w(x)$. Substituting the expression (1.8) for $w(x)$ in equation (1.6), we get

$$\int_0^L EA\frac{du}{dx}\frac{d\phi_i}{dx}dx = \int_0^L f\phi_i dx + P\phi_i\,/_{x=L} \qquad \text{for } i = 1,2,..., N+1. \qquad (1.9)$$

- Now, we write the integral from 0 to L as a sum of the integrals over N elements. Thus, we have

$$\sum_{k=1}^{N}\int_{x_k}^{x_{k+1}} EA\frac{du}{dx}\frac{d\phi_i}{dx}dx = \sum_{k=1}^{N}\int_{x_k}^{x_{k+1}} f\phi_i dx + P\phi_i\,/_{x=L} \qquad \text{for } i =1,2,..., N+1. \qquad (1.11)$$

- Next, to obtain (N +1) algebraic equations for the unknowns u_i, we substitute the approximation (5.5) for $u(x)$ in the above equation. Thus we obtain

$$\sum_{k=1}^{N}\left\{\sum_{j=1}^{N+1}u_j\int_{x_k}^{x_{k+1}} EA\frac{d\phi_j}{dx}\frac{d\phi_i}{dx}dx\right\} = \sum_{k=1}^{N}\left\{\int_{x_k}^{x_{k+1}} f\phi_i dx\right\} + P\phi_i\,/_{x=L} \qquad \text{for } i = 1,2,..., N+1. \qquad (1.12)$$

- Interchanging the sums on the left side, we get

$$\sum_{j=1}^{N+1}u_j\left\{\sum_{k=1}^{N}\int_{x_k}^{x_{k+1}} EA\frac{d\phi_j}{dx}\frac{d\phi_i}{dx}dx\right\} = \sum_{k=1}^{N}\left\{\int_{x_k}^{x_{k+1}} f\phi_i dx\right\} + P\phi_i\,/_{x=L} \qquad \text{for i} = 1,2,.., N+1. \qquad (1.13)$$

- Now, define the following **:**

$$K_{ij} = \sum_{k=1}^{N}\int_{x_k}^{x_{k+1}} EA\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx \qquad \text{for } i = 1,2,.., N+1,\ j = 1,2,.., N+1; \qquad (1.14)$$

-

$$F_i = \sum_{k=1}^{N}\int_{x_k}^{x_{k+1}} f\phi_i dx + P\phi_i\,/_{x=L} \qquad (1.15)$$

for i = 1,2,...., N +1.

- Then, equation (1.13) becomes :

$$\sum_{j=1}^{N+1}K_{ij}u_j = F_i \qquad (1.16)$$

for i = 1,2,.., N +1.

- In matrix form, this can be written as

$$[K]\{u\} = \{F\} \qquad (1.17)$$

- The last term in the expression (5.12) for F_i can be simplified as follows. Note that, at the end point $x = L$, all $\phi_i(x)$ are zero except for $i = N+1$. Therefore, the last term is nonzero only for $i = N+1$. Thus, the expression (1.9) becomes

$$F_i = \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} f\phi_i dx \qquad \text{for } i = 1,2,., N;$$

$$= \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} f\phi_{N+1} dx + P\phi_{N+1}/_{x=L} \qquad \text{for } i = N+1. \qquad (1.18)$$

- The expressions (1.8) and (1.18) show that the stiffness matrix K_{ij} and the force vector F_i are to be evaluated as the sums of the integrals over N elements. The systematic procedure for this evaluation will be discussed in the next Lecture. However, for given i, the contribution of many elements is zero as shown in the next paragraph.
- Consider the expressions (1.8) and (1.18) for K_{ij} and F_i. Note that the functions $\phi_i(x)$, except for i = 1 and $N+1$, are zero outside the interval (x_{i-1}, x_{i+1}). Therefore, the sums (5.11) and (5.15) for K_{ij} and F_i receive the contributions only from the two elements : element i -1 and i, that is, from $\Omega_{i-1} = (x_{i-1}, x_i)$ and $\Omega_i = (x_i, x_{i+1})$ for $i = 2,3,.., N$. For $i = 1$, the function $\phi_i(x)$ is non zero only for the first element $\Omega_1 = (x_1, x_2)$. Thus, for $i = 1$, K_{ij} and F_i receive the contributions only from the first element. For i $= N+1$, the function $\phi_{N+1}(x)$ is non-zero only for the last, i.e., N^{th} element $\Omega_N = (x_N, x_{N+1})$. Thus, for i $= N+1$, K_{ij} and F_i receive the contributions only from the N^{th} element. Thus , the expressions (1.8) and (1.18) for K_{ij} and F_i become :

$$K_{ij} = \underbrace{\int_{x_1}^{x_2} EA \frac{d\phi_1}{dx}\frac{d\phi_j}{dx} dx}_{(1^{st}\text{ element})} \qquad \text{for } i = 1,\ j = 1,2,.., N+1,\ (1^{st}\text{ element})$$

$$= \underbrace{\int_{x_{i-1}}^{x_i} EA \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} dx}_{((i-1)^{th}\text{ element})} + \underbrace{\int_{x_i}^{x_{i+1}} EA \frac{d\phi_i}{dx}\frac{d\phi_j}{dx} dx}_{(i^{th}\text{ element})} \qquad \text{for } i = 2.., N,\ j = 1,2,.., N+1,$$

$$= \underbrace{\int_{x_N}^{x_{N+1}} EA \frac{d\phi_{N+1}}{dx}\frac{d\phi_j}{dx} dx}_{(N^{th}\text{ element})} \qquad \text{for } i = N+1,\ j = 1,2,..,N+1; \qquad (1.19)$$

$$F_i = \underbrace{\int_{x_1}^{x_2} f\phi_1 dx}_{(1^{st}\text{ element})} \qquad \text{for } i = 1,$$

$$= \underbrace{\int_{x_{i-1}}^{x_i} f\phi_i dx}_{((i-1)^{th} element)} + \underbrace{\int_{x_i}^{x_{i+1}} f\phi_i dx}_{(i^{th} element)} \qquad \text{for } i = 2..,N,$$

$$= \underbrace{\int_{x_N}^{x_{N+1}} f\phi_{N+1} dx + P\phi_{N+1}/_{x=L}}_{(N^{th} element)} \qquad \text{for } i = N+1. \qquad (1.20)$$

- It is observed that the stiffness matrix K_{ij} in equation (1.16) is a **sparse matrix** . More precisely, it is a **banded matrix** . This can be shown as follows.
- Note that all $\phi_j(x)$ are zero over the next interval (x_{i-1}, x_i) except when $j = i$ -1 and i . Similarly, all $\phi_j(x)$ are zero over the next interval (x_i, x_{i+1}) except when $j = i$ and i +1. Therefore, the only non-zero contributions to the sum on the left side of the equation (1.16) are from $j = i$ -1, i , i +1. Thus, i^{th} equation in the set (5.13) becomes

$$\sum_{j=i-1}^{i+1} K_{ij} u_j = F_i \qquad (1.21)$$

- where, for i = 2,3,..., N ,

$$K_{ij} = \int_{x_{i-1}}^{x_i} EA \frac{d\phi_i}{dx} \frac{d\phi_{i-1}}{dx} dx \qquad \text{for } j = i-1,$$

$$= \int_{x_{i-1}}^{x_i} EA \frac{d\phi_i}{dx} \frac{d\phi_i}{dx} dx + \int_{x_i}^{x_{i+1}} EA \frac{d\phi_i}{dx} \frac{d\phi_i}{dx} dx \qquad \text{for } j = i,$$

$$= \int_{x_i}^{x_{i+1}} EA \frac{d\phi_i}{dx} \frac{d\phi_{i+1}}{dx} dx \qquad \text{for } j = i+1,$$

$$= 0 \qquad \text{for remaining } j \qquad (1.22)$$

- Further, for i = 1, the only non-zero contributions to the sum on the left side of equation (1.16) come from j=1 and 2. Thus

$$\sum_{j=1}^{2} K_{1j} u_j = F_1 \qquad (1.23)$$

- where

$$K_{1j} = \int_{x_1}^{x_2} EA \frac{d\phi_1}{dx} \frac{d\phi_1}{dx} dx \qquad \text{for } j = 1,$$

$$= \int_{x_1}^{x_2} EA \frac{d\phi_1}{dx} \frac{d\phi_2}{dx} dx \qquad \text{for } j = 2,$$

$$= 0 \qquad \text{for remaining } j. \qquad (1.24)$$

- Similarly, for i = N +1, the only non-zero contributions to the sum on the left side of equation (1.16) come from $j = N$ and N +1. Thus,

$$\sum_{j=N}^{N+1} K_{N+1,j} u_j = F_{N+1} \qquad (1.25)$$

- where

$$K_{N+1,j} = \int_{x_N}^{x_{N+1}} EA \frac{d\phi_{N+1}}{dx} \frac{d\phi_N}{dx} dx \qquad \text{for } j = N,$$

$$= \int_{x_N}^{x_{N+1}} EA \frac{d\phi_{N+1}}{dx} \frac{d\phi_{N+1}}{dx} dx \qquad \text{for } j = N+1,$$

$$= 0 \qquad \text{for remaining } j. \qquad (1.26)$$

- Thus, i^{th} equation has non-zero elements only in the columns j = i-1, i , and i +1. The remaining columns are zero. Further, the 1^{st} equation has non-zero elements only in the 1^{st} and 2^{nd} columns . Similarly, the last equation (i.e., $(N+1)^{th}$ equation) has non-zero elements only in the N^{th} and $(N+1)^{th}$ columns. Thus, the matrix K_{ij} is a banded matrix.

Important Questions

1. Students need to solve the following questions.
   Question no 1.1-1.19, page no. 13 to page no. 16, Segerlind L.J, “Applied Finite Element Analysis”, *John Wiley Publishers, Second Edition, 1976.*

2. Question no 1.1-1.4, page no. 25 to page no. 26, Question no 2.1-2.24, page no. 98 to page no. 102, J. N. Reddy, An Introduction to the Finite Element Method, 3rd Edition.